Algebra, Geometry and Analysis of Commuting Ordinary Differential Operators

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Textbook / Reference

This list of references are compiled for the students of the I International Undergraduate Mathematics Summer School, 2018. It is by no means an extensive list of reference on the subject covered by the class, but just a suggestion for a starting point (or for further reading) with most relevance for this class.

History: First articles about commuting differential operators


Classification of commutative subalgebras of ODOs

Explicit examples of commuting ODOs and explicit determination of the spectral data


Survey articles and books (contain not only the theory of commuting ODOs, but are mainly devoted to various aspects of nonlinear differential equations

- D. Mumford, Tata lectures on Theta II, Birkh¨auser, Boston, 1984
Special lectures for students


Lecture notes

Yes.

Grades

Take-home exam. Exercises will be given after each lecture.

Syllabus

Description

There are two classical problems related to integrable systems, appeared and studied already in the works of I. Schur, J. Burchnall, T. Chaundy in the beginning of 20th century: how to construct explicitly a pair of commuting differential operators and how to classify all commutative subalgebras of differential operators. Both problems have broad connections with many branches of modern mathematics, first of all with integrable systems, since explicit examples of commuting operators provide explicit solutions of many non-linear partial differential equations. The theory of commuting differential operators is far to be complete, but it is well developed for commuting ordinary differential operators.

This course involves an explanation of basic ideas and constructions from the theory of commuting ordinary differential operators as well as an overview of related open problems from algebra, algebraic geometry and complex analysis.

Course topics

The course will cover the following topics:

1. Basic algebraic properties of the ring of ordinary differential operators. Pseudo-differential operators and Schur’s theory.

2. Basic facts and constructions from Commutative Algebra and Algebraic Geometry (needed mostly for algebraic curves): Nullstellensatz, Krull dimension, localisation of rings and modules, discrete valuation rings, smooth points via regular local rings, torsion free modules over regular local rings.

3. Basic algebro-geometric properties of commutative subalgebras of ordinary differential operators. The Burchnall-Chaundy lemma, the notion of algebro-geometric spectral data (spectral curve, spectral bundle).

4. Analytic theory of commuting differential operators: the Baker-Akhieser function, the Jacobian of the spectral curve.

Basic Algebraic Geometry
Wei-Ping Li

Textbook / Reference

1. Griffiths and Harris: Principles of Algebraic Geometry;

2. Griffiths: Introduction to Algebraic Curves.

Lecture notes

Yes.

Grades

Take-home exam.

Syllabus

The content will include the following materials: Concepts and examples of complex manifolds, line bundles, divisors, sheaves and cohomology of sheaves, Riemann surfaces, Riemann-Hurwitz formula, Riemann-Roch formula,
Fiber bundles and Characteristic Classes

Xianzhe Dai

Textbook / Reference

1. *Characteristic classes* by Milnor and Stasheff
2. Witten deformation and Chern-Weil theory by Weiping Zhang
3. Fiber bundles and Chern-Weil theory by Johan Dupont
4. *Differential forms in algebraic topology* by Bott and Tu
6. Geometry, Topology and Physics by M. Nakahara

Lecture notes

Yes.

Grades

50% homework and 50% take-home exam

Syllabus

Characteristic classes are invariants of fiber bundles or principal bundles on manifolds which describes the ‘twisting’ of the bundle (think Möbius band). Besides their importance in the classification of manifolds, they play a crucial role in differential geometry, complex geometry, gauge theory and physics. The goal of this minicourse is to introduce the basic notions of vector bundles and principal bundles, and the theory of characteristic classes, with an emphasis on the geometric aspect and its geometric application.
Harmonic Analysis

Christopher Sogge

Textbook / Reference


Lecture notes

Yes

Grades

Take-home exam

Syllabus

1. A) Classical Harmonic Analysis
   - Fourier transform
   - Distributions
   - Interpolations and multiplier theorems
   - Hardy-Littlewood-Sobolev theorem and elliptic regularity
   - Wave front sets (microlocal analysis of singularities)
   - Oscillatory integral distributions

   If time permits we shall also cover

2. B) Method of stationary phase

3. C) Oscillatory integral estimates and related topics
Symplectic Geometry
Tianjun Li & Bo Dai

Textbook / Reference

1. Dusa McDuff and Dietemar Salamon, Introduction to Symplectic Topology.

Lecture notes
Yes.

Grades
Take-home exam.

Syllabus
Symplectic geometry is the geometry of symplectic manifolds. Two centuries ago, symplectic geometry provided a language for classical mechanics. Through its recent huge development, it conquered an independent and rich territory, as a central branch of differential geometry and topology. Contact geometry is in many ways an odd-dimensional counterpart of symplectic geometry. Both symplectic geometry and contact geometry are motivated by the mathematical formalism of classical mechanics, where one can consider either the even-dimensional phase space of a mechanical system or constant energy hypersurface of odd dimension. The goal of this short course is to provide a fast introduction to symplectic geometry as well as contact geometry. The first week lectures contain basics of symplectic geometry; the second week lectures include basics of contact geometry, and advanced topics on closed symplectic 4-manifolds and symplectic 4-manifolds with contact boundary. The plan of lectures is as follows.

Week 1

1. Lecture 1—Linear symplectic geometry
   Symplectic vector spaces, symplectic linear groups, Lagrangian subspaces, complex structures, Hermitian vector spaces, 4-dimensional geometry

2. Lecture 2—Symplectic manifolds
   Symplectic vector bundles, almost complex structures, examples of symplectic manifolds, spaces of symplectic forms, Moser stability and Darboux theorem, symplectic and Lagrangian submanifolds, neighborhood theorems.

3. Lecture 3—Almost Kahler geometry
   Integrability of almost complex structures, differential calculus on almost Hermitian manifolds, spin$^c$ structures and Dirac operators on 4-manifolds, identities on almost Kahler manifolds.

Week 2
1. Lecture 1—Contact structure
   Liouville vector field, hyper surfaces of contact type, contact structure, symplectic manifolds with contact boundary

2. Lecture 2—Closed symplectic 4-manifolds
   Examples including rational and ruled surfaces, Constructions, Geography, Kodaira dimension, Properties of Seiberg-Witten (little proof for this part)

3. Lecture 3—Symplectic fillings in dimension 4
   Stein domains, Symplectic fillings and caps of contact 3-manifolds, Geography of symplectic fillings,
D. Mumford, â€œAn algebro-geometric construction of commuting operators and of solutions to the Toda lattice equation, Korteweg de Vries equation and related nonlinearequation,â€ in Proceedings of the International Symposium on Algebra and Geometry, Kyoto, 1977 (Kinokuniya, 1978), pp. 115â€“153. Google Scholar. 22. V. Oganesyan, â€œCommuting differential operators of rank 2 with polynomial coefficients,â€ Func. Anal. Appl. V. S. Oganesyan, â€œCommuting differential operators of rank 2 and arbitrary genus $g$ with polynomial coefficients,â€ Russian Math. Surveys, 70:1 (2015), 165â€“167. V. N. Davletshina, â€œSelf-Adjoint Commuting Differential Operators of Rank 2 and Their Deformations Given by Soliton Equations,â€ Math. Notes, 97:3 (2015), 333â€“340. A. B. Zheglov, H. Kurke, â€œGeometric properties of commutative subalgebras of partial differential operators,â€ Sb. A. Mironov A.E. Zheglov A.B., â€œCommuting Ordinary Differential Operators with Polynomial Coefficients and Automorphisms of the First Weyl Algebra,â€ Int. Math. Res. In this paper we study rank two commuting ordinary differential operators with polynomial coefficients and the orbit space of the automorphisms group of the...Moreover, we prove in this case that for any $n \geq 1$ there is a pair of self-adjoint commuting ordinary differential operators of rank two $L_4=(\partial_x^2+V(x))^2+W(x)$, $L_6$, where $W(x),V(x)$ are polynomials of degree $n$ and $n+2$. We also prove that there are hyperelliptic spectral curves with the infinite spaces of orbits. Notes. 15 pages. In particular, invertible ordinary linear differential operators, invertible linear difference operators with periodic coefficients, maps defined by unimodular matrices, and C-transformations of control systems are invertible D-operators. C-Transformations are those invertible transformations for which the variables of one system are expressed in terms of the variables of the other system and their derivatives. In the article we consider the invertible D-operators whose inverses are D-operators of the same type. 6. Chetverikov V.N. Analysis and synthesis of generalized invertible differential operators with one independent variable. Differential Equations, 2015, vol. 51, no. 11, pp. 1529-1539. DOI: 10.1134/S0012266115110142. Entirely elementary methods are employed to determine explicit formulae for the coefficients of commuting ordinary differential operators of orders six and nine which correspond to an elliptic curve. These formulae come from solving the nonlinear ordinary differential equations which are equivalent to the commutativity condition. Most solutions turn out to be rational expressions in one or two arbitrary functions and their derivatives. The corresponding Burchnall-Chaundy curves are computed. Type. Research Article. Information.