CHAPTER 4

QUADRATIC DIOPHANTINE EQUATIONS WITH FOUR UNKNOWNS

This chapter consisting of two sections deals with the problem of obtaining patterns of integral solutions of quadratic Diophantine equations with four unknowns.

Section (A) displays the non- trivial distinct integral solutions of the Diophantine equation

$$z(x+y)+w^2=4xy$$

The non- trivial integer solutions of the equation given below are evaluated in Section (B)

$$2(x+y)+xy=w^2-z^2$$

SECTION- A

We search for non-zero integer quadruples (x, y, z, w) satisfying the

following equation

$$z(x+y)+w^2 = 4xy \tag{4.1}$$

The substitutions

$$x = u+v, y = u-v \tag{4.2}$$

in (4.1) lead to the well-known Space Pythagorean equation

$$(4u-z)^2 = z^2 + (4v)^2 + (2w)^2 \tag{4.3}$$

where u, v are distinct non zero parameters .

Three different patterns of non-trivial integral solutions of (4.1) by applying the solutions to (4.3) are illustrated below.

Pattern I

Using the standard characterizations of (4.3), its non-zero integral solutions are noted as follows

$$z = p^2 - q^2 - r^2$$

$$2v = pq$$

$$w = pr$$

$$4u - z = p^2 + q^2 + r^2$$

Thus,

$$u = \frac{p^2}{2}, \quad v = \frac{pq}{2}$$

According to the aim of finding integer solutions, one obtains

$u = 2P^2, v = Pq$ where $p = 2P$ and q is arbitrary

Substituting the values of u and v in (4.2), the non -zero distinct integral solutions of (4.1) are expressed by

$$x = 2P^2 + Pq$$

$$y = 2P^2 - Pq \quad \text{where} \quad 2P \neq q$$

$$z = 4P^2 - q^2 - r^2$$

$$w = 2\,\text{Pr}$$

Properties

1. $x + y - z$ is written as the sum of two squares.
2. Each of the expression given below is written as difference of two perfect squares.

   (i) $2w$ (ii) $z - \frac{w}{2P} + 2T_r$

3. $x + y - \frac{w}{r} = 2Hex_P$
4. $x + y + \frac{2w}{r} = 8T_P$
5. Each of the following expression represents a perfect square.

   (i) $x + y$

   (ii) $(x + y - z) + \frac{(x - y)}{2P} - 2T_q$

   (iii) $8T_P - \frac{x + y - qw}{P} - z$

   (iv) $\frac{x}{P} - z - 2T_q + 2Hex_P$

   (v) $Dec_P - z + \frac{x + y + qw}{P} - P$

   (vi) $x - 2T_P$ when $q = 1$

6. $\frac{(2q+1)x+(2q-1)y}{2q}-\frac{w}{r}=Oct_P+T_P$

7. $r(x-y)=qw$

8. When $q=1$, $y=Hex_P$

9. When $q=1$, $4y-x=TED_P$

Pattern II

The choice $z=(2w)k$ in (4.3) leads to the equation

$$Y^2=DX^2+M^2 \tag{4.4}$$

where

$$Y=4u-z, D=k^2+1, X=2w, M=4v \tag{4.5}$$

It is obvious that (4.4) is the form of Pythagorean equation which is satisfied by

$$\left.\begin{aligned} X &= 2rs \\ M &= Dr^2-s^2=(k^2+1)r^2-s^2 \\ Y &= Dr^2+s^2=(k^2+1)r^2+s^2 \end{aligned}\right\} \tag{4.6}$$

Employing (4.5), note that

$$u=\frac{(k^2+1)r^2+s^2+2rsk}{4}$$

$$v=\frac{(k^2+1)r^2-s^2}{4}$$

The values of u,v are integers when both r and s are even integers.

Thus, replace r by $2R$ and s by $2S$, we obtain

$$\left.\begin{aligned} u &= (k^2+1)R^2+S^2+2RSk \\ v &= (k^2+1)R^2-S^2 \end{aligned}\right\} \tag{4.7}$$

From (4.5) and (4.6), one gets

$$w = 4RS$$

Substituting (4.7) in (4.2), the non-trivial integral solutions to (4.1) are found to be

$$x = 2(k^2 + 1)R^2 + 2RSk$$

$$y = 2S^2 + 2RSk$$

$$z = 8RSk$$

$$w = 4RS$$

As an immediate consequence the following results are noted

1. Each of the following expression represents a perfect square.
   - (i) $2(4 - zy)$
   - (ii) $2y - wk$
   - (iii) $2(2x - w)$ when $k = 1$
   - (iv) $4x - z$ when $k = 1$
2. $\frac{4(y + w) - z}{8}$ is written as difference of two perfect squares.
3. $3(4y - z)$ is a Nasty number.
4. **When** $R = 1,\ 2y + w - \frac{z}{2} = 8T_S$

Pattern III

Replacing z by $2\bar{z}$ in (4.3), it is written as

$$(2u - \bar{z})^2 = \bar{z}^2 + (2v)^2 + w^2$$

which is satisfied by

$$\bar{z} = p^2 - q^2 - r^2, u = p^2, v = pq, w = 2pr$$

Using the above equations and (4.2), the integral solutions to (4.1) are obtained as

$$x = p^2 + pq$$

$$y = p^2 - pq$$

$$z = 2(p^2 - q^2 - r^2)$$

$$w = 2pr$$

Properties

1. $x + y + \frac{w}{r} = 4T_p$
2. Each of the following expressions represents a perfect square.
   (i) $2\left[Hex_p - Hex_q - (z - \frac{y}{p})\right]$
   (ii) $2\left[OD_p - (8z - 7w) - 16\right]$ when $r = 1$
   (iii) $3\left[12 - 6TED_q - 6z + 5(x - y)\right]$ when $p = 1$
3. $\frac{2(x + y) - z}{2}$ is written as sum of three perfect squares.
4. $Hex_p - z + \frac{w}{2r}$ is written as two times sum of two perfect squares.
5. $z + \frac{w}{2p} + Hex_r$ is written as two times difference of two perfect squares.
6. $\frac{4r(x + y) - 3w}{2r} = Dec_p$
7. $5(x + y) - \frac{4w}{r} = 2DD_p$
8. When $q = 1$, $x + 10y = 2TD_p$
9. When $r = 1$, $9(x + y) - 7w = 4HD_p$
10. When $r = 1$, $9(x + y) - 8w = 4IC_p$

Generation of solutions

Knowing a solution, one may get a sequence of non trivial integral solutions by the procedure given below.

Let (x_0, y_0, z_0, w_0) be the non- zero integral solution to (4.1)

Write $x_1 = x_0 + h, y_1 = y_0 + h, z_1 = z_0 + h, w_1 = h - w_0$ (4.8)

where h is any non zero integer.

Substituting (4.8) in (4.1), we get

$$h = -3x_0 - 3y_0 + 2z_0 - 2w_0 \quad (4.9)$$

Using (4.9) in (4.8), we get

$$x_1 = -2x_0 - 3y_0 + 2z_0 - 2w_0$$
$$y_1 = -3x_0 - 2y_0 + 2z_0 - 2w_0$$
$$z_1 = -3x_0 - 3y_0 + 3z_0 - 2w_0$$
$$w_1 = -3x_0 - 3y_0 + 2z_0 - 3w_0$$

which is represented in the matrix form as follows

$$(x_1 \quad y_1 \quad z_1 \quad w_1)^T = M(x_0 \quad y_0 \quad z_0 \quad w_0)^T$$

where M is the 4×4 matrix $\begin{pmatrix} -2 & -3 & 2 & -2 \\ -3 & -2 & 2 & -2 \\ -3 & -3 & 3 & -2 \\ -3 & -3 & 2 & -3 \end{pmatrix}$ and T refers to the transpose.

The repetition of the above process leads to the general solutions represented as below

$$\begin{pmatrix} x_{s+1} \\ y_{s+1} \\ z_{s+1} \\ w_{s+1} \end{pmatrix} = \begin{pmatrix} \frac{3\tilde{y}_s + 1}{4} & \frac{3\tilde{y}_s - 3}{4} & \frac{-\tilde{y}_s + 1}{2} & \tilde{x}_s \\ \frac{3\tilde{y}_s - 3}{4} & \frac{3\tilde{y}_s + 1}{4} & \frac{-\tilde{y}_s + 1}{2} & \tilde{x}_s \\ \frac{3\tilde{y}_s - 3}{4} & \frac{3\tilde{y}_s - 3}{4} & \frac{-\tilde{y}_s + 3}{2} & \tilde{x}_s \\ \frac{3\tilde{x}_s}{2} & \frac{3\tilde{x}_s}{2} & -\tilde{x}_s & \tilde{y}_s \end{pmatrix} \begin{pmatrix} x_0 \\ y_0 \\ z_0 \\ w_0 \end{pmatrix}$$

where $\tilde{y}_s + \sqrt{2}\tilde{x}_s = (-3 - 2\sqrt{2})^{s+1}$, $s = 0,1,2,3,......$

Thus, knowing (x_0, y_0, z_0, w_0) and substituting $s = 0,1,2,3,......$ in turn one generates an infinitely many integral solutions.

SECTION -B

Consider the pairs of non-zero integers given by $(x, y) = (6,1)$ and $(5,2)$. It is noted that each pair satisfies the relation $2(x + y) + xy = w^2 - z^2$.

In what follows,we exhibit infinitely many rectangles satisfying the equation

$$2(x + y) + xy = w^2 - z^2 \qquad (4.10)$$

where x, y represent the dimensions of the rectangle and x, y, w, z are non-zero integers.

We present below three different methods of solving (4.10).

Method 1

Introducing the linear transformations

$$x = u + v, \quad y = u - v, \quad w = p + q, \quad z = p - q \qquad (4.11)$$

where $u, v, p, q \neq 0$, $u \neq v$, $p \neq q$ in equation (4.10), it is written as

$$(u + 2)^2 - v^2 = 4(pq + 1)$$

Again setting $pq = \alpha^2 - 1$, the above equation becomes

$$(u + 2)^2 - v^2 = (2\alpha)^2 \qquad (4.12)$$

Equation (4.12) represents the well-known Pythagorean equation.

Employing the standard solutions of this equation, one gets

$$u + 2 = r^2 + s^2$$

$$v = r^2 - s^2, \quad r \neq s$$

$$\alpha = rs$$

Applying (4.11), the non-zero integral solutions of (4.10) are given by

$$x = 2r^2 - 2$$

$$y = 2s^2 - 2$$

$$w = p + q$$

$$z = p - q$$

where $r > s > 1$ and p, q are chosen such that $pq = r^2 s^2 - 1$.

Some numerical examples are exhibited as follows

Table 4.1(a)

r	s	p	q	x	y	w	z
3	2	7	5	16	6	12	2
4	3	143	1	30	16	144	142
4	2	63	1	30	6	64	62
5	3	112	2	48	16	60	52
		56	4	48	16	114	110
		28	8	48	16	36	20
		16	14	48	16	30	2

A few interesting properties observed from the solutions are given below

1. $x - y$ is a perfect square for the following choices of r and s given in Table 4.1(b)

Table 4.1(b)

r	s
$2m^2 + n^2$	$2m^2 + n^2$
$m^2 + 2n^2$	$m^2 + 2n^2$

2. $x + y + 4$ is a perfect square for the following choices of r and s given in the following Table 4.1(c)

Table 4.1(c)

r	s
$m^2 - n^2 + 2mn$	$m^2 - n^2 - 2mn$
	$2mn - m^2 + n^2$

3. when $r^2 = 3\alpha^2 + 1$, x is a Nasty number. A similar holds for y.

4. If $r = 2^n$, then $x = 6J_{2n}$

   Also, $x = 2(j_{2n} - 2)$

Method 2

Choosing

$$x = u + v,\ y = u - v,\ w = \alpha^2,\ z = \alpha^2 - 2,\ u, v \neq 0, u \neq v \tag{4.13}$$

in equation (4.10), it is written as

$$(u+2)^2 - v^2 = 4\alpha^2 \tag{4.14}$$

Setting

$$u + 2 = 2\tilde{u} \text{ and } v = 2\tilde{v} \tag{4.15}$$

in (4.14), one obtains

$$\tilde{u}^2 - \tilde{v}^2 = \alpha^2 \tag{4.16}$$

Equation (4.16) is the well known Pythagorean equation.

Hence, employing the most cited solutions of it, one gets

$$\tilde{u} = r^2 + s^2,\quad \tilde{v} = 2rs,\quad \alpha = r^2 - s^2$$

In view of (4.15), it is to be noticed that

$$u = 2(r^2 + s^2) - 2,\quad v = 4rs$$

Therefore, the non –trivial integral solutions of (4.10) are found to be

$$x = 4r^2 - 2$$

$$y = 4s^2 - 2$$

$$w = 4r^2 s^2$$

$$z = 4r^2 s^2 - 2$$

where r and s are non- zero distinct positive integers.

Some numerical examples are given in the following Table 4.1(d)

Table 4.1 (d)

r	s	x	y	w	z
2	1	14	2	16	14
3	2	34	14	144	142
4	3	62	34	576	574
5	4	98	62	1600	1598
6	5	142	98	3600	3598
7	6	194	142	7056	7054
8	7	254	194	12544	12542
9	8	322	254	20736	20734
10	9	398	322	32400	32398

A few interesting properties observed from the solutions are presented below

1. The following expressions are represented a perfect square respectively.
   - (i) $x+y+4\alpha-3$ when $s=2$, $r=2+\alpha$
   - (ii) $x+y+\alpha^2$ when $s=1+\alpha$, $r=2+\alpha$
   - (iii) $x+y+8\alpha-4$ when $s=3$, $r=3+\alpha$
   - (iv) $x-y$ when $r=p^2+q^2$, $s=p^2-q^2$ or $2pq$ $(p>q>1)$
   - (v) $4w-xy+4$ when $r=p^2-q^2+2pq$, $s=p^2-q^2-2pq$ or $2pq-p^2+q^2$ $(p>q>1)$

2. Denoting the solutions x and y by the notations $x(r)$ and $y(s)$ respectively, the recurrence relations satisfied by the solutions are illustrated below

   (i) $x(r+1)-y(s)+16=x(r+2)-y(s+1)$

   (ii) $x(r)-y(s)+8=x(r+1)-y(s+1)$

Remark

It is to be noticed that, for this case we have another set of solutions given as below.

$$x = 2(r^2 + s^2) + 4rs - 2$$
$$y = 2(r^2 + s^2) - 4rs - 2$$
$$w = (r^2 - s^2)^2$$
$$z = (r^2 - s^2)^2 - 2$$

Method 3

Substituting $x = z$, (4.10) is rewritten as

$$z^2 + z(y+2) + 2y - w^2 = 0 \qquad (4.17)$$

Treating this equation as a quadratic in z and solving for z we get

$$z = \frac{1}{2}[-(y-2) + \sqrt{(y-2)^2 + 4w^2}]$$

The square root on the R .H.S of the above equation is eliminated on substituting

$$y = r^2 - s^2 + 2, \; w = rs \qquad (4.18)$$

and thus,

$$x = z = s^2 - 2 \qquad (4.19)$$

The values of x, y and w given in (4.18) and (4.19) represent the solutions of (4.10).

Properties

1. Each of the following expression represents a perfect square.
   (i) $x + y$ (ii) $2(x + w + 1) + y$
2. when $r = 1$, $z + w + 2 = 2T_s$
3. when $s = 1$, $y + w + x = 2T_r$
4. If $s = 2^n$, then $x = z = 3J_{2n} - 1$.Also, $x = z = j_{2n} - 3$.

Solution of Algebraic and Transcendental Equations- Introduction: The Bisection. Method â€“ The Method of False Position â€“ The Iteration Method - Newton â€“Raphson. Unit-IV Solution of Non-linear Systems.Â etc. In order to solve above type of equations following methods exist. Directive Methods: The methods which are used to find solutions of given equations in the direct process is called as directive methods. Example: Synthetic division, remainder theorem, Factorization method etc Note: By using Directive Methods, it is possible to find exact solutions of the given equation. Equations like this are called transcendental equations. Euler-buckling load for a fixed-pinned beam. Y = a Cosh (x/c), equation for a catenary. Solutions to these equations are always obtained iteratively. Starting point is really important for obtaining the proper solution. Lot of insight can be obtained from geometry and pictures. Example: Natural Freqencies of cantilever. There are different ways of approaching a non-linear problem. m is called the order of convergence. Note that for most of these methods the upper and lower bound for the root [a,b] has to be given it is called as bracketi Solution of Algebraic and transcendental equations. The equation of the form f(x) =0 are called algebraic equations if f(x) is purely a polynomial in x.For example: are algebraic equations . If f(x) also contains trigonometric, logarithmic, exponential function etc. then the equation is known as transcendental equation. Methods for solving the equation. The following result helps us to locate the interval in which the roots of.Â Exercise: 1. Determine the real root of correct to four decimal places by Regula-Falsi method. Ans: 1.0499. 2. Find the positive real root of correct to four decimals by the method of False position . Ans: 1.8955. 3.Solve the equation by Regula-Falsi method,correct to 4 decimal places. How do I solve the transcendental equation as shown below for 'th'?. All the other variables (z0, tp, v1, y0, x0 and theta01) are known. However the equation is part of a code and the variables (z0, tp, v1, y0, x0 and theta01) are inputs of a function and it will be chosen by the user. So, whenever they input any value for those 6 variables the 'th' must be the output. I've tried to bring only 'th' to one side of the equation but I think is impossible. Note: I'm new in Matlab. So, please, try to be as clear as possible.Â You have a quadratic, so there will be two solutions. The equation is not transcendental with respect to the variable of interest, th. 2 Comments. ShowHide all comments. (ii) Every algebraic equation of nth degree has n and only n real or imaginary roots. Conversely, if Î±1, Î±2,â€¦â€¦, Î±n be the n roots of the nth degree equation f (x) = 0, then f (x) = A(x â€“ Î±1)(x â€“ a2)â€¦â€¦â€¦(x â€“ Î±n). (iii) If f (x) is continuous in the interval [a, b] and f (a), f (b) have different signs, then the equation have at least one root between.Â 4 Solutions of Algebraic and Transcedental Equations 965. P = Point of intersection of two curves. y(x).Â This method of solving transcendental equations, consists in locating the roots of the equation f (x) = 0 between two numbers say a and b such that f (x) is continuous for a â‰¤ x â‰¤ b and f (a) and f (b) are of opposite signs so that the product f (a) f (b) < 0 i.e. the curve cuts the x-axis. between a and b. Then the desired root is approximately (b, f (b)) a+b x1 = .