Composition Operators and Isometries on Holomorphic Function Spaces over Domains in $\mathbb{C}^n$

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1 Introduction

Let D be a bounded domain in $\mathbb{C}^n$ with C^1 boundary. Many holomorphic function spaces over D have been introduced in last half century, such as Hardy, Bergman, Besov and Sobolev spaces. Properties (boundary behaviour ect.) of functions in those function spaces have been received a great deal of studies. Readers can see parts of them from the following books [26] [29], [66], [71], [75], [76] and references therein. For any $0 < p \leq \infty$, we let $L^p(\partial D)$ be Lebesgue $p-$space with respect to Lebesgue surface measure on ∂D, let $H^p(D)$ be Hardy space of holomorphic functions on D with their boundary value functions belonging to $L^p(\partial D)$. Let $L^p(D)$ be the Lebesgue p-space with respect to the Lebesgue volume measure on D, and let $A^p(D)$ be its holomorphic subspace. It is well-known that $H^2(D)$, $L^2(\partial D)$, $L^2(D)$ and $A^2(D)$ are Hilbert spaces; $H^2(D)$ is a closed subspace of $L^2(\partial D)$ while $A^2(D)$ is a closed subspace of $L^2(D)$. There are two important orthogonal projection operators: $S : L^2(\partial D) \to H^2(D)$ and $P : L^2(D) \to A^2(D)$. S is called Szegö projection and P is called Bergman projection. Both of them can be written as integral operators:

$$(1.1) \qquad Sf(z) = \int_{\partial D} S(z, w)f(w)d\sigma(w); \quad Pg(z) = \int_D K(z, w)f(w)dv(w), \quad z \in D$$

where $S(z, w)$ is Szegö kernel and $K(z, w)$ is the Bergman kernel for D; they are holomorphic in z and anti-holomorphic in w. If one let M_f be the multiplication operator defined as $M_f u = fu$, then the commutator between M_f with S and P are given by $[M_f, S]$ and $[M_f, P]$. Toepliz operator associated with S and P are $T_f = SM_f$ and $T_f = PM_f$. Characterizations for f so that those operators are bounded, compact, or belonging to the Schattern von Neumann class have been obtained by many authors in different function spaces([2], [15], [14], [8], [9], [36], [37], etc. and reference therein).

Another interesting and important class of operators is composition operators. Let $\phi : D \to D$ be a holomorphic map. Then the composition operator associated to ϕ is defined as $C_\phi(u)(z) = u(\phi(z))$ for any function u on D. In the past three decades, a great deal of researches have been done on composition operators on some function spaces over D. Most of the works have been focussed on finding characterizations of ϕ so that C_ϕ is boundeded, compact, or in a Schatten class on some function spaces such as the Hardy spaces $\mathcal{H}^p(D)$, Bergman spaces $A^p(D)$, Dirichlet spaces and many others (see the book of Cowen and MacCluer [23], the survey paper of Russo [67], the survey paper of Wogen [78], Li [46] and the references therein.) When $\phi : D \to D$ is a biholomorphic map, one expects that C_ϕ is a very special operator on some function spacea. Let $D = B_1$ be the unit disk in complex plane $\mathbb{C}$. Then the holomorphic Besov space is: $B^2(B_1) = \{f : \|f\|_{B^2} < \infty\}$ with $\|f\|^2_{B^2} = \int_{B_1} |f'(z)|^2 dA(z)$. One can easily show that C_ϕ is an isometry on $B^2(B_1)$ with the semi norm $\|\cdot\|_{B^2}$ for any biholomorphic map or Möbuis transformation ϕ from B_1 to itself. One will see later, C_ϕ is an isometry on many function spaces when $\phi \in \mathrm{Aut}(D)$, the automorphism group of D. On the other hands, one expects that an isometry on many function spaces should be very special. In fact, characterizations for an isometry on Hardy space, Bergman space and little Block space have been obtained by several authors (see [21], [70], [41], [39, 40], [38] and references therein).

The main purpose of this note is:

1) Give a brief survey on some results of composition operators, which are closely related to works of the author and his collaborators, by viewing composition operators as special Toeplitz operators.

2) Provide several ways to define function spaces so that C_ϕ is an isometry on those function spaces for any $\phi \in \mathrm{Aut}(D)$, and give the equivalent relations between those spaces and the some well-known function spaces.

3) We will introduce some interesting known results on the charaterizations for isometries on some function spaces, and pose some questions for further studies.

2 Composition Operators

It is well-known that C_ϕ is bounded on any holomorphic Hardy space $\mathcal{H}^p(D)$ when $D \subset \mathbb{C}$ is bounded domain with C^1 boundary for all $0 < p \leq \infty$. In fact, for any $f \in H^p(D)$, one can solve the Dirichlet boundary value problem of Laplacian Δ:

$$\Delta u = 0 \text{ in } D; \text{ and } u(z) = |f(z)|^p, \; z \in \partial D. \tag{2.1}$$

Since $|f(z)|^p$ is subharmonic, maximum principle shows that $|f(z)|^p \leq u(z)$ on D. Since $C_\phi(u)$ solves (2.1) with boundary condition $u = |C_\phi(f)|^p$. Therefore,

$$|C_\phi(f)|^p \leq C_\phi(u) \text{ in } D. \tag{2.2.}$$

Let $P(z, w)$ be the Poisson kernel on $D \times \partial D$ for Dirichlet problem (2.1). For any fixed point $z_0 \in D$, there are two constants $0 < a(z_0) \leq A(z_0) < \infty$, depending only on D and z_0, so that $a(z_0) \leq P(z_0, w) \leq A(z_0)$ for $w \in \partial D$. Thus

$$\begin{aligned} \int_{\partial D} |C_\phi(f)|^p d\sigma(z) &\leq \int_{\partial D} u(\phi(z)) d\sigma(z) \\ &\leq \frac{1}{a(z_0)} \int_{\partial D} u(\phi(z)) P(z_0, z) d\sigma(z) \end{aligned}$$

$$
\begin{aligned}
&= \frac{u(\phi(z_0))}{a(z_0)} \\
&= \frac{1}{a(z_0)} \int_{\partial D} u(z) P(\phi(z_0), z) d\sigma(z) \\
&\leq \frac{A(\phi(z_0))}{a(z_0)} \int_{\partial D} |f(z)|^p d\sigma(z).
\end{aligned}
$$

Therefore,

$$
\|C_\phi(f)\|_{H^p(D)} \leq \Big[\frac{A(\phi(z_0))}{a(z_0)}\Big]^{1/p} \|f\|_{H^p(D)}. \tag{2.3}
$$

In particular, when $D = B_1$ the unit disk. By choosing $z_0 = 0$, then $a(0) = A(0) = 1$ and $A(z) \leq \frac{1+|z|}{1-|z|}$. Hence

$$
\|C_\phi(f)\|_{H^p(B_1)} \leq \Big[\frac{1+|\phi(0)|}{1-|\phi(0)|}\Big]^{1/p} \|f\|_{H^p(B_1)}. \tag{2.4}
$$

The above argument is no longer true when $n > 1$ since $C_\phi(u)$ may not be harmonic in D if u is merely harmonic in D when $n > 1$. In fact, counterexamples were constructed in [20] and [58] to show that there is a holomorphic map $\phi = (2z_1z_2, 0) : B_2 \to B_2$ so that C_ϕ is neither bounded on $\mathcal{H}^p(B_2)$ nor $A^p(B_2)$ for all $0 < p < \infty$, where B_n is the unit ball in $\mathbb{C}^n$. Let $\phi : D \to D$ be a holomorphic map, we define two pull-bak measures v_ϕ^0 and v_ϕ as follows: For any measurable set $E \subset D$, we let

$$
v_\phi^0(E) = \sigma_{\partial D}(\phi^{-1}(E) \cap \partial D\}), \quad v_\phi(E) = v(\phi^{-1}(E)) \tag{2.5}
$$

where $\phi^{-1}(E) = \{z \in \overline{D} : \phi(z) \in E\}$. Then one has the following theorem (see [23] and [49], [28], [8], etc. for details.)

THEOREM 2.1 *Let D be a bounded strictly pseudoconvex domain in $\mathbb{C}^n$ with C^1 boundary. Let $\phi : D \to D$ be a holomorphic map. Then*

(i) C_ϕ is bounded on $H^p(D)$ if and only if v_ϕ^0 is a Carleson measure;

(ii) C_ϕ is compact if and only if v_ϕ^0 has vaninshing Carleson measure.

Similar theorem holds for Bergman spaces.

If one let

$$
B_{\phi,0}(z)^2 = S(z,z)^{-1} \int_D |S(z,w)|^2 dv_\phi^0(w), \quad B_\phi(z)^2 = K(z)^{-1} \int_D |K(z,w)|^2 dv_\phi(w) \tag{2.6}
$$

Then Theorem 2.1 can be restated as

THEOREM 2.2 *Let D be a bounded strictly pseudoconvex domain in $\mathbb{C}^n$ with C^∞ boundary. Let $\phi : D \to D$ be a holomorphic map. Then*

(i) C_ϕ is bounded on $H^p(D)$ if and only if $B_{\phi,0} \in L^\infty(D)$;

(ii) C_ϕ is compact if and only if $B_{\phi,0} \in C_0(D)$.

Similar theorem holds for Bergman spaces by replacing $B_{\phi,0}$ by $B_\phi(z)$.

Compact composition operators on $A^2(B_1)$ were first characterized by Shapiro [72] by using Nevanlinna counting function. Hilbert-Schmidt and nuclear composition operators on $\mathcal{H}^2(B_1)$ were characterized by Shapiro and Taylor in [73]. Leucking and Zhu in [52] also used the Nevanlinna counting function to characterize Schatten class composition operators on $\mathcal{H}^2(B_1)$ and $A^2(B_1)$. When D is a smoothly bounded strictly pseudoconvex domain, the following result was proved by the author [44]:

THEOREM 2.3 *Let D be a smoothly bounded strictly pseudoconvex domain in $\mathbb{C}^n$. Then $C_\phi \in S_p(A^2(D))$ (Schatten Von Neumann $p-$class) if and only if $X_p(\phi) < \infty$, where*

$$X_p(\phi)^p = \int_D \left[K(z,z)^{-1} \int_D |K(z,\phi(w))|^2 dv(w)\right]^{p/2} d\lambda(z) \tag{2.6}$$

for $\frac{2n}{n+1} < p < \infty$, where $d\lambda(z) = K(z,z)dv(z)$.

When D is bounded symmetric domain in $\mathbb{C}^n$, Schattern p-class compositions were characterized by the author and B. Russo [49] by the pull-back measure dv_ϕ for $0 < p < \infty$. It is always important and interesting to find a simple and natural condition on the map ϕ which characterizes compact, and Shatten class composition operators C_ϕ. In [58], MacCluer and Shapiro proved the following: *C_ϕ is compact on $A^2(B_1)$ if and only if $K(z,z)^{-1}K(\phi(z),\phi(z)) \to 0$ as $z \to \partial B_1$.* In [52], Leucking and Zhu proved that: If $D = B_1$ then $Y_p(\phi) \geq \|C_\phi\|_{S_p(A^2(D)}$ for $0 < p \leq 2$; and $\|C_\phi\|_{S_p(A^2(D))} \geq Y_p(\phi)$ for $2 \leq p < \infty$. Here

$$Y_p(\phi)^p = \int_D \left[K(\phi(z),\phi(z))K(z,z)^{-1}\right]^{p/2} d\lambda(z). \tag{2.7}$$

One may extend the latter result to any smoothly bounded domain in $\mathbb{C}^n$ by using their argument (see [44].) A natural question was posed in [52]. Problem: *Let $2 \leq p < \infty$ and $D = B_1$, is it true $C_\phi \in S_p(A^2(D))$ if and only if $Y_p(\phi) < \infty$?*

Partial results were given in [44], in which the author proved that

$$\frac{1}{C(\phi,D)} Y_p(\phi) \leq \|C_\phi\|^p_{S_p(A^2(D))} \leq C(\phi,D)Y_p(\phi), \tag{2.8}$$

where D is a smoothly bounded strictly pseudoconvex domain in $\mathbb{C}^n$ and $C(\phi,D)$ is a constant depending only on D, $\|\phi\|_{C^{n+1}(\overline{D})}$ and $\|C_\phi\|_{S_\infty}$. The smoothness assumption on ϕ here is somewhat natural for $n > 1$ because we know that C_ϕ is not bounded on $A^2(B_2)$ where $\phi(z_1,z_1) = (2z_1z_2, 0)$, a polynomial, but it is not natural for the case $n = 1$. Recently, K. Zhu [**?**] posed another condition for the case $n = 1$. He proved that *(1.3) holds with $C(\phi,B_1) = C(N)$, a positive constant depending on $N = \sup\{\#(\phi^{-1}(w)) : w \in B_1\} < \infty$. Here $\#(E)$ denotes the number of elements in E.* It is clear that the quantity (2.7) is simpler than the quantity (2.6) which characterizes the Shatten class composition operators. It is puzzling and interesting to find out whether $X_p(\phi)$ and $Y_p(\phi)$ are equivalent when $p \neq 2$. In other words, whether the constant $C(N)$ (in the result of Zhu) does really depend on N. In [45] (1999, preprint), the author proves that the quantities (1.1) and (1.2) are not equivalent when $D = B_1$ and $p \neq 2$, which answers the question in [52] negatively in the sense of the equivalent norms. A precise example was constucted by Jingbo, Xia [79], which shows that there is $\phi \in Y_p(B_1)$, but $\|C_\phi\|_{S_p} = \infty$ for $2 < p < \infty$.

Let

$$Z(\phi)^{2p} = \int_D \Big[\int_D |K(\phi(z), \phi(w))|^{p'} dv(w) \Big]^{\frac{p}{p'}} dv(z) \tag{2.9}$$

and

$$W_p^p(\phi) = \int_D \Big[\delta(z)^{n+2} \int_{\partial D} |K(z, \phi(w))|^2 d\sigma(w) \Big]^{p/2} d\lambda(z). \tag{2.10}$$

Another criteria for Schatten class composition operators on $A^2(D)$ in terms of $Z(\phi)$ and on Harday space interms of $W(\phi)$ was also obtained in [45], which can be stated as follows:

criterion for Schatten composition operators on $\mathcal{H}^2(D)$.

THEOREM 2.4 *Let D be a smoothly bounded strictly pseudoconvex domain in $\mathbb{C}^n$. Let $\phi : D \to D$ be a holomorphic map.*

(i) For $2 \le p \le \infty$, we have

$$\frac{1}{C_p} W_p(\phi) \le \|C_\phi\|_{S_p(\mathcal{H}^2(D))} \le C_{p,q} W_q(\phi) \tag{2.11}$$

for any $q < p$;

(ii) Let $1 \le p < \frac{2n}{n-1}$ and let D be either a smoothly bounded strictly pseudoconvex domain or a bounded symmetric domain in $\mathbb{C}^n$. Then

$$\frac{1}{C_p} Z_{2p}(\phi) \le \|C_\phi\|_{S_{2p}(A^2)} \le C_p Z_{2p}(\phi). \tag{2.12}$$

where C_p is a positive constant depending only on p and D.

There are many other researches on characterizations for boundedness, compactness of composition on weighted Bergman spaces from A^p to A^q, or other function spaces, we mention a few examples here. For examples, we refer the readers to papers of Z. Cuckovic and R. Zhao [16], Luo and Shi and Zhou [74], Zhao [80], etc. and references therein.

3 Isometric Composition Operators

For a bounded domain D in $\mathbb{C}^n$, we let $\mathrm{Aut}(D)$ be the automorphism group of D. In this section, we will define some function spaces, on which the composition operator C_ϕ is an isometry for any $\phi \in \mathrm{Aut}(D)$. Let

$$B^2(D) = \Big\{ f \in A^2(D) : \|f\|_{B^2}^2 = \int_D |\partial f(z)|^2 dv(z) < \infty \Big\}. \tag{3.1}$$

When D is a bounded domain in the complex plane, it is easy to verify that C_ϕ is an isometry on $B^2(D)$ for any $\phi \in \mathrm{Aut}(D)$. This is no longer true when $n > 1$. It has been proved by

Arazy, Fisher, Janson and Peetre [3], K. Zhu [82] and Peloso [63] that C_ϕ is an isomorphism on $B^2(D)$ when D is the unit ball in $\mathbb{C}^n$ and $\phi \in \mathrm{Aut}(D)$. The results for more general function spaces over a more general domains were obtained by the author and Luo [47]. We will provide a straight forward way to define a general holomorphic Besov spaces so that

(a) C_ϕ is an isometry on it for any $\phi \in \mathrm{Aut}(D)$;

(b) The new norm is equivalent to the usual norm.

Let $\delta(z)$ be the distance function from z to ∂D. For any $0 < p < \infty$, Besov space over a domain D consisting of all holomorphic functions f on D so that

$$\|f\|^p_{B^p(D)} = \int_D |\partial f(z)|^p \delta(z)^p d\lambda(z) < \infty. \tag{3.2}$$

It is easy prove that $B^p = \mathbb{C}$ when $p \le n$ since $p - n - 1 \le -1$ and $\int_D \delta(z)^{-1} dv(z) = \infty$. In order to avoid B^p becomes trivial when $0 < p \le n$, one may change the definition for B^p by replacing $|\partial f(z)|\delta(z)$ by $|\partial^k f(z)|\delta(z)^k$ for $pk > n$ (see [47] for details). In general, C_ϕ is not an isometry on $B^p(D)$ when $n > 1$. So we introduce here some new norms for new Besov spaces, and prove they are equivalent to old ones.

First, we use a Kähler metric on D to define holomorphic function spaces. Let $g = g_{\alpha\overline{\beta}} dz_\alpha \otimes d\overline{z}^\beta$ be a Kähler metric over D. Then the Laplace operator associated to g is

$$\Delta_g = \sum_{\alpha,\beta=1}^n g^{\alpha\overline{\beta}}(z) \frac{\partial^2}{\partial z_\alpha \partial \overline{z}_\beta}, \quad |\nabla f|^2_g = \sum_{\alpha,\beta} g^{\alpha\overline{\beta}} \left(\frac{\partial f}{\partial z_\alpha} \overline{\frac{\partial f}{\partial z_\beta}} + \frac{\partial \overline{f}}{\partial z_\alpha} \overline{\frac{\partial \overline{f}}{\partial z_\beta}}\right) \tag{3.3}$$

Let $\mathcal{O}(D)$ be the set of all holomorphic functions over D. For any $0 < p < \infty$, we let $B^p_g(D)$ be the set of all functions $f \in \mathcal{O}(D)$ so that

$$\|f\|^p_{g,p} = \int_D |\nabla f(z)|^p_g dv_g(z) < \infty, \quad dv_g(z) = |\det(g_{\alpha\overline{\beta}})|^2 dv(z). \tag{3.4}$$

As we know from a result of Bell [10] and Boas and Straube [13] that if D is a smoothly bounded domain in $\mathbb{C}^n$ with a smooth plurisubharmonic defining function then ϕ can be extended smoothly up to the boundary of D for any $\phi \in \mathrm{Aut}(D)$. If g is a metric so that there is a constant $C \ge 1$ with

$$\frac{1}{C} g(z) \le g(z) \circ \phi(z) \le C g(z) \tag{3.5}$$

Then C_ϕ is an isomorphic on $B^p_g(D)$ for each $\phi \in \mathrm{Aut}(D)$.

Further more, if g is an ivariant metric, i.e., g is either Bergman metric or Kähler-Einstein metric, then C_ϕ is an isometry on $B^p_g(D)$ for any $\phi \in \mathrm{Aut}(D)$. In fact, for those two metrics, we have

$$dv_g(z) = dv_g(\phi(z)), \quad \sum_{\alpha,\beta=1}^n g^{\alpha\overline{\beta}}(z) \frac{\partial \phi_k}{\partial z_\alpha} \overline{\frac{\partial \phi_\ell}{dz_\beta}} = g^{k\overline{\ell}}(\phi(z)). \tag{3.6}$$

This implies that

$$\Delta_g |C_\phi f(z)|^2 = (\Delta_g |f|^2)(\phi(z)). \tag{3.7}$$

Notice that if f is holomorphic, then

$$|\nabla f|_g^2 = \Delta_g |f(z)|^2. \tag{3.8}$$

Thus

$$\|C_\phi(f)\|_{B_g^p} = \|f\|_{B_g^p}. \tag{3.9}$$

When D is a smoothly bounded strictly pseudoconvex domain in $\mathbb{C}^n$, one may use the result of C. Fefferman [24] on asympototic expansion for Bergman kernel function, Cheng and Yau [19], Lee and Melrose [42] on global regularity for the potential function of Kähler-Einstein metric, as well as the analysis of E. Stein [76], to prove $B_g^p(D)$ and $B^p(D)$ are equivalent when $p > 2n$.

Second, we use the Bergman kernel function to define function spaces $Y_p(D)$, which are equivalent to Besov spaces in many cases. Those spaces $Y_p(D)$ were also used a lot in studying the theory of Hankel operators, commutators (see for examples, [2], [3], [8], [9], [14] and [43], ect. and references therein.) For each $z \in D$, we let

$$k_z(w) = K(z,z)^{-1/2} K(w,z), \ \ w \in D. \tag{3.10}$$

For any $f \in A^2(D)$ we let (Berezin transform)

$$B(f)(z)^2 = \int_D |f(w) - f(z)|^2 |k_z(w)|^2 dv(w). \tag{3.11}$$

The function space $Y_p(D)$ is defined as follows:

$$Y_p(D) = \{f \in A^2(D) : \|f\|_{Y_p(D)} < \infty\}, \quad \|f\|_{Y_p(D)}^p = \int_D B(f)^p(z) d\lambda(z). \tag{3.12}$$

For any $\phi \in \mathrm{Aut}(D)$, since $|k_z(w)|^2 = |k_{C_\phi(z)}(\phi(w)|^2 |\det \phi'(w)|^2$, we have

$$\begin{aligned} B(C_\phi(f))(z)^2 &= \int_D |C_\phi(f)(w) - C_\phi(f)(z)|^2 |k_z(w)|^2 dv(w) \\ &= \int_D |C_\phi(f)(w) - C_\phi(f)(z)|^2 |k_{C_\phi(z)}(C_\phi(w))|^2 |\det \phi'(w)|^2 dv(w) \\ &= \int_D |f(w) - C_\phi(f)(z)|^2 |k_{C_\phi(z)}(w)|^2 dv(w) \\ &= B(f)^2(\phi(z)) \end{aligned} \tag{3.13}$$

Since $d\lambda(z)$ is invariant measure under biholomorphic map, we have

$$\|f\|_{Y_p} = \|C_\phi(f)\|_{Y_p(D)}, \qquad \text{for all } \phi \in \mathrm{Aut}(D). \tag{3.14}$$

Therefore C_ϕ is an isometry on $Y_p(D)$ for all $\phi \in \mathrm{Aut}(D)$.

Third, for each $0 < p \le \infty$, one can easily see that C_ϕ is an isometry on $L^p(D, d\lambda)$ with norm:

$$\|h\|_{L^p(D,d\lambda)} = \Big[\int_D |h(z)|^p d\lambda(z)\Big]^{1/p}. \tag{3.15}$$

The relations between $B^p(D)$, $B_g^p(D)$ and $Y_p(D)$ can be described as the following theorem.

THEOREM 3.1 *Let D be a smoothly bounded strictly pseudoconvex domain in $\mathbb{C}^n$. Then*
(i) If $0 < p \le n$ then $B^p(D) = B^p_g(D) = Y_p(D) = \mathbb{C}$;
(ii) If $n < p \le 2n$ then $B^p_g(D) = Y_p(D) = \mathbb{C}$, and $B^p(D) \ne \mathbb{C}$ when $p > n$;
(iii) If $2n < p < \infty$, there is a constant $C_p > 0$ so that

$$\frac{1}{C_p}\|f\|_{B^p(D)} \le \|f\|_{B^p_g(D)} \approx \|f\|_{Y_p(D)} \le C_p\|f\|_{B^p(D)} \tag{3.16}$$

for all $f \in B^p(D)$.

Remark: The above theorem remains true on pseudoconvex domain of finite type domain in $\mathbb{C}^2$ or convex domain of finite type in $\mathbb{C}^n$ by replacing 2 (strictly pseudocovex domain is type 2) by type m (see, Li and Luo [47] for details of some arguments.)

There is another space, a little bit different from $B^\infty(B)$ (Bloch spac), is BMOA(D). If one defines BMOA-norm (semi-norm) as:

$$A(f)(z)^2 = \int_{\partial D} |f(w) - f(z)|^2 \frac{|S(z,w)|^2}{S(z,z)} d\sigma(w), \tag{3.17}$$

and

$$\|f\|_{BMOA(D)} = \|A(f)\|_{L^\infty(D)}. \tag{3.18}$$

Then for any $f \in H^2(D)$, we have that for $BMO(\partial D)$-norm for f is equivalent to usual BMOA norm as (3.18) when D is strictly pseudoconvex domain in $\mathbb{C}^n$. With the definition (3.18), we have the following theorem.

THEOREM 3.2 *Let D be a bounded symmetric domain in $\mathbb{C}^n$. Then C_ϕ is an isometry on $BMOA(D)$ for any $\phi \in Aut(D)$.*

Proof. Let $\phi_z \in \mathrm{Aut}(D)$ so that $\phi_z(0) = z$ and $\phi_z \circ \phi_z = I$. Then

$$\phi_{z_0} \circ \phi_z(w) = \phi_{\phi_{z_0}(\phi_z(0))}(w)$$

and (see, for example, [68])

$$\int_{\partial D} g(\phi_z)(w) d\sigma(w) = \int_{\partial D} g(w) P(z,w) d\sigma(w), \quad \phi_z \in \mathrm{Aut}(D)$$

$$\begin{aligned} \|f\|^2_{BMOA(D)} &= \sup\Big\{ \int_{\partial D} |f(w) - f(z)|^2 P(z,w) d\sigma(w) : z \in D \Big\} \\ &= \sup\Big\{ \int_{\partial D} |f \circ \phi_z(w) - f \circ \phi_z(0)|^2 d\sigma(w) : \phi_z \in \mathrm{Aut}(D) \Big\} \end{aligned}$$

Let $\phi_{z_0} \in \mathrm{Aut}(D)$ so that $\phi_{z_0}(0) = z_0$. Then

$$\begin{aligned}
&\|f \circ \phi_{z_0}\|^2_{BMOA(D)} \\
&= \sup\Big\{ \int_{\partial D} |(f \circ \phi_{z_0}) \circ \phi_z(w) - (f \circ \phi_{z_0})(\phi_z(0))|^2 d\sigma(w) : \phi_z \in \mathrm{Aut}(D) \Big\} \\
&= \sup\Big\{ \int_{\partial D} |f \circ \phi_{\phi_0(\phi_z(0))}(w) - f \circ \phi_{\phi_0(\phi_z(0))}(0)|^2 d\sigma(w) : \phi_z \in \mathrm{Aut}(D) \Big\} \\
&= \sup\Big\{ \int_{\partial D} |f \circ \phi_z(w) - f \circ \phi_z(0)|^2 d\sigma(w) : \phi_z \in \mathrm{Aut}(D) \Big\} \\
&= \|f\|^2_{BMOA}.
\end{aligned}$$

Therefore, the proof of the theorem is complete. □

Now what is the relation between $L^p(D, d\lambda)$ and $B^p(D)$? The problem was stduied by Zhu [83] for bounded symmetric domain, Li and Luo [47] for more general pseudoconvex domain in $\mathbb{C}^n$.

Let $d\Lambda(z) = K(z,z)^{-1} dv(z)$ with K being the Bergman kernel function. Let $\tilde{K}$ be the Bergman kernel on D with respect to the measure $d\Lambda(z)$. We define an operator

$$V(f)(z) = K(z)^{-1} \int_D \tilde{K}(z,w) f(w) dv(w). \tag{3.19}$$

Then

$$PV = I_n \text{ on } A^2(D). \tag{3.20}$$

In fact,

$$\begin{aligned}
PV(f)(z) &= \int_D K(z,w) V(f)(w) dv(w) \\
&= \int_D K(z,w) K(w,w)^{-1} \int_D \tilde{K}(w,\xi) f(\xi) dv(\xi) dv(w) \\
&= \int_D \int_D K(z,w) K(w,w)^{-1} \tilde{K}(w,\xi) dv(w) f(\xi) dv(\xi) \\
&= \int_D K(z,\xi) f(\xi) dv(\xi) \\
&= f(z).
\end{aligned}$$

It was proved in [47] the following theorem.

THEOREM 3.3 *Let D be a smoothly bounded strictly pseudoconvex in $\mathbb{C}^n$ or pseudoconvex domain of finite type in $\mathbb{C}^n$ or convex domain of finite type in $\mathbb{C}^n$. Then $V : B^p(D) \to L^p(D, d\lambda)$ is bounded. Moreover, $P : L^p(D, d\lambda) \to B^p(D)$ is bounded.*

Note: The above theorem was only true when $n < p \le \infty$ for usual definition for B^p, it remains true for general p if one modifies the definition for B^p when $p \le n$.

For any $\phi \in \mathrm{Aut}(D)$, it is easy to show that

$$\tilde{K}(z,w) = \tilde{K}(\phi(z), \phi(w)) (\det \phi'(z))^2 \overline{\det \phi'(w)}^2 \tag{3.21}$$

If we let

$$E(\phi)(w) = \frac{\det \phi'(w)}{\overline{\det \phi'(w)}}, \tag{3.22}$$

then

Proposition 3.4 *For any biholomorphic map $\phi : D \to D$, we have*

$$VC_\phi(f)(z) = E_\phi(z)\, C_\phi V(PM_{E(\phi^{-1})} f). \tag{3.23}$$

Note that

$$\begin{aligned}
V(f\circ\phi)(z) &= K(z,z)^{-1}\int_D \tilde{K}(z,w) f(\phi(w)) dv(w) \\
&= K(z,z)^{-1}(\det\phi'(z))^2 \int_D \tilde{K}(\phi(z),\phi(w)) f(\phi(w)) \overline{\det\phi'(w)}^2 dv(w) \\
&= E(\phi)(z) K(\phi(z),\phi(z))^{-1}\int_D \tilde{K}(\phi(z),w) f(w) E(\phi^{-1})(w) dv(w) \\
&= E(\phi)(z) K(\phi(z),\phi(z))^{-1}\int_D \tilde{K}(\phi(z),w) P(f(w)E(\phi^{-1}))(w) dv(w) \\
&= E(\phi)(z) C_\phi(V(P(fE(\phi^{-1}))))
\end{aligned}$$

and

$$\|V(f\circ\phi)\|^p_{L^p(D,d\lambda)} = \|E(\phi)V(fE(\phi))(\phi)\|^p_{L^p(D,d\lambda)} = \|V(P(fE(\phi)))\|^p_{L^p(D,d\lambda)}. \tag{3.24}$$

Therefore, when $\phi \in \mathrm{Aut}(D)$ with $\phi \in C^1(\overline{D})$, then $PM_{E(\phi^{-1})} : B^p(D) \to B^p(D)$ is bounded. One can see from Theorem 3.3, (3.23) and (3.24) that C_ϕ is an isomorphism on $B^p(D)$ since C_ϕ is an isometry on $L^p(D, d\lambda)$.

4 Isometries

Characterizing an isometry on a given function spaces has been received some studies, some known results along these directions can be summarized as follows:

1) Let D be either the unit ball or unit polydisk in $\mathbb{C}^n$. If $T : H^p(D) \to H^p(D)$ with $p \in (1,2) \cup (2.\infty]$ is an onto isometry, then

$$T(f)(z) = e^{i\theta}\mathcal{P}(\phi^{-1}(0), z)^{-1/p} C_\phi(f)(z)$$

for some $\phi \in \mathrm{Aut}(D)$, where $\mathcal{P}(z,w) = S(z,z)^{-1}|S(z,w)|^2$ is Poisson-Szegö kernel.

When D is the unit disk, Statement 1) was proved by Nagasawa in 1959 for $p = \infty$, by deLeeuw, Rudin and Wermer for $p = 1$ in 1960 and by Forelli [25] for all p.

When D is polydisc, Statement 1) was proved by Schneider in 1971; and when D is the unit ball, Statement 1) was proved by Forelli for $p > 2$ and Rudin [70] for general p.

The following theorem was proved by Koranyi and Vagi [41]

2) Let D be a bounded symmetric domain in $\mathbb{C}^n$. If $T : H^p(D) \to H^p(D)$ with $p \in (1,2) \cup (2.\infty]$ is an into isometry, then

$$T(f)(z) = T(1)C_\phi(f)(z)$$

for some $\phi \in \mathrm{Aut}(D)$.

For Bergman spaces, the following theorem was proved by Kolaski [39] and [40].

3) Let D be a bounded Runge domain in $\mathbb{C}^n$. Let T be an into isometry on $A^p(D)$ for $p \in (1,) \cup (2,\infty)$. Then $T(f)(z) = T(1)(z)C_\phi(f)(z)$ for some $\phi \in \mathrm{Aut}(D)$.

For little Bloch space, Cima and Wogen [21] for $n = 1$ and Krantz and Ma [38] for all n prove the following result:

4) Let D be the unit ball in $\mathbb{C}^n$. Let T be an isometry on $\mathcal{B}_0(D)$. Then $T(f)(z) = \mu[C_\phi(f)(z) - C_\phi(f)(0)]$ for some $\phi \in \mathrm{Aut}(D)$.

When D is polydisk, some result related 4) was obtained by L. Chen in [18].

Viewing above four results, and C_ϕ is an isometry on those Besov space, BMOA constructed in Section 3. One may pose the following question.

Problem: Let D be a smoothly bounded strictly pseudoconvex domain or polydisk in $\mathbb{C}^n$. Let X be one of the following function spaces: $X = Y_p(D)$, $\tilde{B}^p(D)$, $BMOA(D)$, $VMOA(D)$, and let T is an isometry on X. Is it true $Tf = aC_\phi f - bC_\phi(f)(z_0)$ for some $\phi \in Aut(D)$, complex numbers a, b with $|a| = 1$ and some point $z_0 \in D$?

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In this survey, we present a development of the theory of multiplication operators and composition operators from classical spaces of analytic functions H(G) to the weighted spaces of analytic functions HVb(G,E) and HV0(G,E). References. J. C. Evard and F. Jafari, â€œOn semigroups of operators on Hardy spaces,â€ preprint, 1995.Â T. Mazur, â€œCanonical isometry on weighted Bergman spaces,â€ Pacific Journal of Mathematics, vol. 136, no. 2, pp. 303â€“310, 1989. View at: Google Scholar | Zentralblatt MATH | MathSciNet.Â D. GarcÃa, M. Maestre, and P. Sevilla-Peris, â€œComposition operators between weighted spaces of holomorphic functions on Banach spaces,â€ Annales AcademiÃ¦ Scientiarium FennicÃ¦. Mathematica, vol. 29, no. 1, pp. 81â€“98, 2004. In mathematics, a holomorphic function is a complex-valued function of one or more complex variables that is, at every point of its domain, complex differentiable in a neighborhood of the point. The existence of a complex derivative in a neighbourhood is a very strong condition, for it implies that any holomorphic function is actually infinitely differentiable and equal, locally, to its own Taylor series (analytic). Holomorphic functions are the central objects of study in complex analysis. Though the term analytic function is often used interchangeably with "holomorphic function", t Then the holomorphic functions and agree on the common domain and are hence restrictions of a single entire function . But goes to zero at infinity and is hence bounded; applying Liouvilleâ€™s theorem (Theorem 28 of Notes 3) we see that vanishes entirely. This gives on and on .Â Two meromorphic functions , are said to be equivalent if they agree on their common domain of definition . It is easy to see that this is an equivalence relation. It is common to identify meromorphic functions up to equivalence, similarly to how in measure theory it is common to identify functions which agree almost everywhere. Exercise 9 (Meromorphic functions form a field) Let denote the space of meromorphic functions on a connected open set , up to equivalence. We characterise continuity of composition operators on weighted spaces of holomorphic functions Hv(BX), where BX is the open unit ball of a Banach space which is homogeneous, that is, a JBâˆ—âˆ'triple. 1 Introduction. In this note, we prove a result concerning composition operators on JBâˆ—âˆ'triples.Â The rst case considered was that of B being the unit disc or a domain in C or Cn. Special interest has been given to the study of composition operators between these spaces; we refer to [6, 8, 9] and particularly to the recent surveys [5, 7] and the references therein for infor-mation about the subject. Some study has also been devoted to the situation when BX is the open unit ball of a Banach space X (see e.g. [3, 13, 14]).